AMERICAN Journal of Language, Literacy and Learning in STEM Education
Volume 02, Issue 04, 2024 ISSN (E): 2993-2769

The Existence of Odd Perfect Numbers

Zafar Karimov
Student of the Physics and Mathematics Faculty of Termez State University

Abstract: This paper is devoted to the history of the problem of existence of odd perfect numbers, the oldest unsolved mathematical problem.

Keywords: number, perfect number, prime, odd, equal, problem, sum, divisors, integer, triangle, binary, sigma function, divisors function, Marsenne primes.

INTRODUCTION

The problem of the existence of odd perfect numbers is the oldest unsolved mathematical problem, which is already 2000 years old. Some of the brightest mathematicians of all time like Nicomachus, Fermat, Marsenne and Euler have tried to solve it, but all of them failed. Solving this problem can be as simple as finding a single number.

In the middle of the 20th century mathematicians have used computers and checked numbers up to 10 to the power of 2200, but so far they have been unsuccessful. This problem has captured the imaginations of so many mathematicians because it's old, it's simple and it's beautiful.

PERFECT NUMBERS

Let’s take the number six. We can divide it by 1, 2 and 3. If we add up all these divisors, we find that they add up to six, which is the number itself. Numbers like this are perfect. We can also try this with other numbers like 24. 24 has the divisors 1, 2, 3, 4, 6, 8, and 12. If we add those up, we only get 36. So 24 is not a perfect number. Now we repeat this for all the other numbers from 1 to 100 and see that only 6 and 28 are perfect. Go up to 10000 and we find the next two perfect numbers 496 and 8128. These were the only perfect numbers known by the ancient Greeks, and they would be the only known ones for over a thousand years.

$$6$$
$$28$$
$$496$$
$$8128$$

Consider the general features of perfect numbers. One thing to notice is that each next perfect number is one digit longer than the number that came before it. Another thing they share is that the ending digit alternates between 6 and 8, which also means they are all even. But here's where things get really weird. We can write 6 as the sum of 1 plus 2 plus 3 and 28 as the sum of one, plus 2, plus 3, plus 4 plus 5 plus 6 plus 7, and so on for the others as well, they're all just the sum of consecutive numbers.

$$6 = 1 + 2 + 3,$$
$$28 = 1 + 2 + 3 + 4 + 5 + 6 + 7,$$
$$496 = 1 + 2 + 3 + 4 + 5 + \cdots + 30 + 31,$$
$$8128 = 1 + 2 + 3 + 4 + 5 + 6 + 7 + \cdots + 126 + 127;$$

And so these create a triangle, which is why these numbers are called triangular numbers.

Also, every number except for six is the sum of consecutive odd cubes. 28 is 1 cubed plus 3 cubed. 496 is equal to 1 cubed plus 3 cubed plus 5 cubed plus 7 cubed. And 8128 is equal to 1 cubed plus 3 cubed plus 5 cubed plus 7 cubed plus 9 cubed all the way up to 15 cubed.

$$28 = 1^3 + 3^3,$$

$$496 = 1^3 + 3^3 + 5^3 + 7^3,$$

$$8128 = 1^3 + 3^3 + 5^3 + 7^3 + 9^3 + \cdots + 15^3;$$

If we write these numbers in binary, 6 becomes 110, and 28 becomes 11100. 496 becomes 111110000. And 8128 becomes 1111111000000. It is also a string of ones followed by a series of zeros.

$$6_{10} = 110_2,$$

$$28_{10} = 11100_2,$$

$$496_{10} = 111110000_2,$$

$$8128_{10} = 1111111000000_2;$$

And we can also write them out so that they are all just consecutive powers of two.

$$6 = 2^2 + 2^1,$$

$$28 = 2^4 + 2^3 + 2^2,$$

$$496 = 2^8 + 2^7 + 2^6 + 2^5 + 2^4,$$

$$8128 = 2^{12} + 2^{11} + 2^{10} + 2^9 + 2^8 + 2^7 + 2^6;$$

So a ***perfect number*** (Greek: ἀριθμὸς τέλειος) is a positive integer that is equal to the sum of all its own divisors (that is, all positive divisors other than the number itself).

THE HISTORY

What now around 300 BC Euclid discovered the pattern that makes these perfect numbers. He took the number one and double it, he got two, keep doubled it and get 4, 8, 16, 32, 64, and so on.

$$1, 2, 4, 8, 16, 32, 64, 128, \cdots$$

He started from one and added the next number to it. If that added up to a prime, then he multiplied it by the last number in the sequence to get a perfect number. That is:

$$1 + 2 = 3,$$

$$2 \cdot 3 = 6$$

Thus two times three equals six, the first perfect number. And so Euclid found a way to generate even perfect numbers, but he didn't prove that this was the only way. And formula that gives a perfect number is

$$\mathbf{(2^p - 1) \cdot 2^p} .$$

400 years later, the Greek philosopher Nicomachus published *«Introdutio Arithmetica»*, the standard arithmetic textfor the next thousand years. In it, he stated five conjectures statements he believed to be true, but did not bother actually trying to prove. His conjectures were:

1. The *n*th perfect number has *n* digits.
2. All perfect numbers are even.
3. All perfect numbers end in 6 and 8 alternately.
4. Euclid algorithm produces every even perfect number.
5. There are infinitely many perfect numbers.

For the next thousand years no one could prove or disprove any of these conjectures, and they were considered facts. But in the 13th century, Egyptian mathematician *Ibn Fallus* published a list with 10 perfect numbers and their values of p. Three of these perfect numbers turned out not to be perfect at all.

p	Perfect number $(2^p - 1) \cdot 2^p$
2	6
3	28
5	496
7	8 128
9	130 816
11	2 096 128
13	33 550 336
17	8 589 869 056
19	137 438 691 328
23	35 184 367 894 528

But the remaining ones are. The fifth perfect number is eight digits long, which disproves Nicomachus's first conjecture. And the next thing to notice is that both the fifth and sixth perfect number end in a 6. So that disproves Nicomachus's third conjecture that all perfect numbers end in a 6 or 8 alternately. Two conjectures were proven false.

Two centuries later, the problem reached Renaissance Europe where they rediscovered the fifth, sixth, and seventh perfect numbers. So far every perfect number had Euclid's form. And the best way to find new ones was by finding the values of p that make $(2^p - 1)$ prime. So French polymath Marin Mersenne extensively studied numbers of this form. In 1644, he published his in a book including a list of 11 values of p for which he claimed they corresponded to primes.

Numbers for which this is true are now called Mersenne Primes. Of his list the first seven exponents of p do result in primes and they correspond to the first seven perfect numbers. But for some of the larger numbers like $2^{67} - 1$, Mersenne admitted to not even checking whether they were prime. He was saying: *"To tell if a given number of 15 to 20 digits is prime or not all time would not suffice for the test".*

Mersenne discussed the problem of perfect numbers with other mathematicians of the time, including Pierre de Fermat and Rene Descartes. In 1638, Descartes wrote to Mersenne, *"I think I can show that there are no even perfect numbers except those of Euclid"*. He also believed that if an odd perfect number does exist, it must have a special form. It must be the product of a prime and the square of a different number, that is $N = p_s M^2$.

If he was right, these would easily have been the biggest breakthroughs on the problem since Euclid 2000 years earlier. But Descartes couldn't prove either of those statements. Instead, he wrote *"As for me, I judge that one can find real odd perfect numbers. But whatever method you use, it takes a long time to look for these".*

Around a hundred years later at the St.Petersburg Academy, the Prussian mathematician Christian Goldbach met a 20-year-old math prodigy. The two stayed in touch corresponding by mail, and in 1729, Goldbach introduced this young man to the work of Fermat. At first, he seemed indifferent, but after a little more prodding by Goldbach he became passionate about number theory and he spent the next 40 years working on different problems in the field among them was the problem of perfect numbers. This Prodigy's name was *Leonhard Euler*.

Euler picked up where Descartes had left off, but with more success. In doing so, he made three breakthroughs on this problem. First in 1732, he discovered the eighth perfect number, which he had done by verifying that $(2^{31} - 1)$ is prime. Just as Mersenne had predicted. For his other two breakthroughs, he invented the *sigma function*.

The functions $d(n)$ and $\sigma_k(n)$.

The function $d(n)$ is the number of divisors of *n*, including 1 and *n*, while $\sigma_k(n)$ is the sum of the *k*th power of the divisors of *n*. Thus

$$\sigma_k(n) = \sum_{d|n} d^k,\ d(n) = \sum_{d|n} l\,.$$

If $n = p_1^{a_1} p_2^{a_2} \cdots p_l^{a_l}$, then the divisors of n are the number $p_1^{b_1} p_2^{b_2} \cdots p_l^{b_l}$, where $0 \le b_1 \le a_1$, $0 \le b_2 \le a_2$, $\cdots$ $0 \le b_1 \le a_l$.

There are $(a_1 + 1)(a_2 + 1) \cdots (a_l + 1)$ of these numbers. Hence

Theorem 1:

$$d(n) = \prod_{i=1}^{l} (a_i + 1).$$

More generally, if $k > 0$,

$$\sigma_k(n) = \sum_{b_1}^{a_1} \sum_{b_2}^{a_2} \cdots \sum_{b_i}^{a_l} p_1^{b_1 k} p_2^{b_2 k} \cdots p_l^{b_l k} = \prod_{i=1}^{l} (1 + p_i^k + p_i^{2k} + \cdots + p_i^{a_i k}).$$

Hence

Theorem 2:

$$\sigma_k(n) = \prod_{i=1}^{l} \left(\frac{p_i^{(a_i+1)k} - 1}{p_i^k - 1} \right).$$

With his new function in hand, Euler achieved his second breakthrough and did what Descartes couldn't. He proved that every even perfect number has Euclid's form. This Euclid − Euler theorem solved a 1600-year-old problem and proved Nicomachus's fourth conjecture. But Euler wasn't done yet. He also wanted to solve the problem of odd perfect numbers. So for his third breakthrough, he set out to prove Descartes other statement that every odd perfect number must have a specific form. But he could not prove this. Euler wrote: *"Whether there are any odd perfect numbers is the most difficult question"*.

For the next 150 years very little progress was made and no new perfect numbers were discovered. English mathematician Peter Barlow wrote that *"Euler's eighth perfect number is the greatest that ever will be discovered for as they are merely curious without being useful, it is not likely that any person will ever attempt to find one beyond it"*. But Barlow was wrong. Mathematicians kept pursuing perfect numbers and most started with Mersenne's primes list.

The next on his list was $2^{67} - 1$. But in 1876 Edouard Lucas the French mathematician proved that $2^{67} - 1$ was not prime, although he was unable to find its factors. 27 years later, on October 31, 1903 Frank Nelson Cole the American mathematician gave a talk to the American mathematical society without saying a word, he walked to one side of the blackboard and wrote down

$$2^{67} - 1 = 147.573.952.589.676.412.927\,.$$

He then walked to the other side of the blackboard and multiplied 193.707.721 times 761.838.257.287 giving the same answer. He sat down without saying a word and the audience erupted in applause. He later admitted it took him three years working on Sundays to solve this. A modern computer could solve this in a milliseconds.

From 500 BC until 1952 people had discovered just 12 Mersenne primes and therefore only 12 perfect numbers. The main difficulty was checking whether large Mersenne numbers were actually prime. But in 1952, American mathematician Raphael Robinson wrote a computer program to perform this task and he ran it on the fastest computer at the time, the SWAC. Within 10 months, he found the next five Mersenne primes and so corresponding perfect numbers. And over the next 50 years, new Mersenne primes were discovered in rapid succession, all using computers. The largest Mersenne prime at the end of 1952 was $2^{2281} - 1$, which is 687 digits long. By the end of 1994, the largest Mersenne prime was $2^{859433} - 1$, which is 258.716 digits long. Nowadays there are 51 Mersenne primes. In 2017 Church Deacon John Pace discovered the 50th Mersenne Prime by using GIMPS.

The **Great Internet Mersenne Prime Search (GIMPS)** is a collaborative project of volunteers who use freely available software to search for *Mersenne prime* numbers. GIMPS was founded in 1996 by George Woltman.

The number $2^{77232917} - 1$ is more than 23 million digits long, and it was also the largest known prime at the time. A year later, the 51st Mersenne Prime was discovered. It's $2^{82589933} - 1$, and this number has 24.862.048 digits. The Mersenne primes are just so large and rare that they take a lot of time and computer resources to find.

CONCLUSION

To this day, the problem of odd perfect numbers is considered to be the oldest unsolved mathematical problem. For more than 2000 years, number theory had no practical application. Mathematicians simply followed their curiosity and solved problems that seemed interesting to them, proving one result after another and creating the foundation of today's mathematics.

List of references:

1. G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Number* (Fourth Edition, Oxford, 1975).
2. *I. D. Mordukhai – Boltovsky, The ninth book of Euclid's Elements (Russian), (Moscow, 1949)*
3. Джавлиева Г.Р. *On The Importance Of Historicism Elements In Mathematics Lessons In Elementary Grades* //The American Journal of Applied Sciences (Vol.15.09.2020).C.1-10.
4. Джавлиева Г.Р. *Интерактивные методы в обучении математике* // JOURNAL OF NEW CENTURY INNOVATIONS. Volume – 5_ Issue-1_May_2022.C.30-35.