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Isometries of *log* -Integrable Functions

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Abstract: In this paper studied isometries of $F-$spaces of integrable functions with logarithm. In the paper, it is given isomorphic classification of F-spaces of log-integrable measurable functions constructed using different measures. At the same time, it is proved that such spaces are non-isometric.

Keywords: Functional spaces; boolean algebras; complete boolean algebras; homogeneous Boolean algebras; internal $\log-$algebras; external $\log-$algebras; generalized $\log-$algebras; isometries.

1. Introduction

One of the important classes of Banach functional spaces are spaces $L_p(\Omega, \mathrm{A}, \mu)$, $1 \le p < \infty$ of all p-th power integrable measurable functions given on the measurable space $(\Omega, \mathrm{A}, \mu)$ with the finite measure μ (almost everywhere equal functions are identified). The study of isometries of Banach spaces L_p was initiated by S. Banach [1], who gave a description of all isometries for spaces $L_p[0,1]$, $p \neq 2$. In [2], J. Lamperti gave characterization of all linear isometries for L_p-spaces $L_p(\Omega, \mathrm{A}, \mu)$ where $(\Omega, \mathrm{A}, \mu)$ is an arbitrary space with the finite measure μ. The final result in this setting is due to Yeadon [3] who gave a complete description of all isometries between L_p-spaces associated with different measures. One of the corollaries of such descriptions of isometries in spaces $L_p(\Omega, \mathrm{A}, \mu)$ is the establishment of isometry for L_p-spaces $L_p(\Omega, \mathrm{A}, \mu)$ and $L_p(\Omega, \mathrm{A}, \nu)$ in the case when the measures μ and ν are strictly positive finite measures.

An important metrizable analogue of Banach spaces L_p are F-spaces $L_{\log}(\Omega, \mathrm{A}, \mu)$ of log-integrable measurable functions introduced in the work [4]. An F-space $L_{\log}(\Omega, \mathrm{A}, \mu)$ is defined by the equality

$$L_{\log}(\Omega, \mathrm{A}, \mu) = \{ f \in L_0(\Omega, \mathrm{A}, \mu) : \int_{\Omega} \log(1+|f|) d\mu < +\infty \}$$

where $L_0(\Omega, \mathrm{A}, \mu)$ is the algebra of all measurable functions given on $(\Omega, \mathrm{A}, \mu)$ (almost everywhere equal functions are identified). By virtue of the inequality

$$\log(f(\omega)) \leq \frac{1}{p} f(\omega)^p, \omega \in \Omega, \ p \in [1, \infty),$$

the inclusion $L_p(\Omega, \mathrm{A}, \mu) \subset L_{log}(\Omega, \mathrm{A}, \mu)$ is always true.

In [4], it was established that $L_{log}(\Omega, \mathrm{A}, \mu)$ is a subalgebra in the algebra $L_0(\Omega, \mathrm{A}, \mu)$. In addition, a special F-metric $\rho(f, g)$ has been introduced in $L_{log}(\Omega, \mathrm{A}, \mu)$,

$$\rho(f, g) = \int_\Omega \log(1 + | f - g |) d\mu, \ f, g \in L_{\log}(\Omega, \mathrm{A}, \mu).$$

The pair $(L_{\log}(\Omega, \mathrm{A}, \mu), \rho)$ is a complete metric topological vector space with respect to this measure, and the operation of multiplication $f \cdot g$ is continuous in the totality of variables.

It is evident, algebras $L_0(\Omega, \mathrm{A}, \mu)$ and $L_0(\Omega, \mathrm{A}, \nu)$ coincide when the measures μ and ν are strictly positive finite measures. This fact is no longer true for the algebras $L_{log}(\Omega, \mathrm{A}, \mu)$ and $L_{log}(\Omega, \mathrm{A}, \nu)$. In the work [5], it was shown that $L_{log}(\Omega, \mathrm{A}, \mu) = L_{log}(\Omega, \mathrm{A}, \nu)$ for strictly positive finite measures μ and ν if and only if $\frac{d\nu}{d\mu} \in L_\infty(\Omega, \mathrm{A}, \mu)$ and $\frac{d\mu}{d\nu} \in L_\infty(\Omega, \mathrm{A}, \nu)$, where $L_\infty(\Omega, \mathrm{A}, \mu)$ is the algebra of all essentially bounded measurable functions given on $(\Omega, \mathrm{A}, \mu)$ (almost everywhere equal functions are identified), $\frac{d\nu}{d\mu}$ (respectively, $\frac{d\mu}{d\nu}$) is the Radon-Nikodym derivation of the measure ν (respectively, μ) with respect to the measure μ (respectively, ν).

Isometries on these F-spaces were considered in [6]. In these papers a description of isometries on F-spaces was given. In contrast to these results, in this paper we establish a necessary and sufficient condition for the existence of isometries and isomorphisms of algebras of log-integrable functions constructed by different measures μ and ν. The relationship between these isometries and isomorphisms is also studied. In this case, conditions are imposed only on the Radon-Nikodym derivatives $\frac{d\nu}{d\mu}$.

Naturally the problem arises to find the necessary and sufficient conditions providing an isomorphism of the algebras $L_{log}(\Omega, \mathrm{A}, \mu)$ and $L_{log}(\Omega, \mathrm{A}, \nu)$ for the strictly positive finite measures μ and ν. The solution of this problem is given in section 4.

The main purpose of this paper is to prove the absence of surjective isometries from $L_{log}(\Omega, \mathrm{A}, \mu)$ onto $L_{log}(\Omega, \mathrm{A}, \nu)$ in the case when measures μ and ν are strictly positive finite measures (see section 4).

2. Preliminaries

Let $(\Omega, \mathrm{A}, \mu)$ be a be a complete measure space with finite measure μ and let $L_0(\Omega, \mathrm{A}, \mu)$ (respectively, $L_\infty(\Omega, \mathrm{A}, \mu)$) be the algebra of equivalence classes of real valued measurable functions (respectively, essentially bounded real valued measurable functions) on (Ω, A, μ). Denote by $\nabla = \nabla_\mu$ the complete Boolean algebra of all equivalence classes $e = [A]$, $A \in A$, of equal μ-almost everywhere sets from the σ-algebra A. It is known that $\mu(e) = \mu([A]) = \mu(A)$ is a strictly positive finite measure on ∇_μ. In what follows, we also denote the measure μ by μ, and the algebra $L_0(\Omega, A, \mu)$ (respectively, $L_\infty(\Omega, A, \mu)$) by $L_0(\nabla) = L_0(\nabla_\mu)$ (respectively, $L_\infty(\nabla) = L_\infty(\nabla_\mu)$).

Following to [4], consider in $L_0(\nabla_\mu)$ a subalgebra

$$L_{\log}(\nabla_\mu) = \{ f \in L_0(\nabla_\mu) : \int_\Omega \log(1 + | f |) d\mu < +\infty \}$$

of log-integrable measurable functions, and for each $f \in L_{\log}(\nabla_\mu)$, set

$$\| f \|_{\log} = \int_\Omega \log(1 + | f |) d\mu.$$

By [4, Lemma 2.1], a nonnegative function $\| \cdot \|_{\log} : L_{\log}(\nabla_\mu) \to [0, \infty)$ is a F-norm, that is,

(i). $\| f \|_{\log} > 0$ for all $0 \neq f \in L_{\log}(\nabla_\mu)$;

(ii). $\| \alpha f \|_{\log} \leq \| f \|_{\log}$ for all $f \in L_{\log}(\nabla_\mu)$ and real number α with $| \alpha | \leq 1$;

(iii). $\lim_{\alpha \to 0} \| \alpha f \|_{\log} = 0$ for all $f \in L_{\log}(\nabla_\mu)$;

(iv). $\| f + g \|_{\log} \leq \| f \|_{\log} + \| g \|_{\log}$ for all $f, g \in L_{\log}(\nabla_\mu)$.

In [4] it is shown that $L_{\log}(\nabla_\mu)$ is a complete topological algebra with respect to the topology generated by the metric $\rho(f, g) = \| f - g \|_{\log}$.

Let μ and ν be two strictly positive measures on the measurable space (Ω, A). Then

$$L_0(\nabla_\mu) = L_0(\nabla_\nu) = L_0(\nabla), \ L_\infty(\nabla_\mu) = L_\infty(\nabla_\nu) = L_\infty(\nabla).$$

Let $\dfrac{d\nu}{d\mu}$ be the Radon-Nikodym derivative of measure ν with respect to the measure μ. It is well known that $0 \leq \dfrac{d\nu}{d\mu} \in L_0(\nabla)$ and

$$f \in L_1(\Omega, A, \nu) \Leftrightarrow f \cdot \frac{d\nu}{d\mu} \in L_1(\Omega, A, \mu),$$

in addition,

$$\int_\Omega f \, d\nu = \int_\Omega f \cdot \left(\frac{d\nu}{d\mu}\right) d\mu.$$

Note that for strictly positive measures μ and ν, it follows that $\left(\dfrac{d\nu}{d\mu}\right)^{-1} = \dfrac{d\mu}{d\nu}$.

If measure ν is finite, then $\frac{d\nu}{d\mu} \in L_1(\nabla_\mu)$. This follows from the equality $\mu(h) = \nu(\mathbf{1}) < \infty$.

3 Isometries of the F-spaces $L_{\log}(\nabla_\mu)$ and $L_{\log}(\nabla_\nu)$

In this section, a necessary and sufficient condition for the existence of isometries is established of $L_{log}(\Omega, \mathrm{A}, \mu)$ onto $L_{log}(\Omega, \mathrm{A}, \nu)$.

Let μ and ν strictly positive finite measures, $h = \frac{d\nu}{d\mu}$, and let

$$L_p(\nabla_\mu) = \{ f \in L_0(\nabla_\mu) : \mathrm{P} f \mathrm{P}_p = (\int_\Omega | f |^p \, d\mu)^{\frac{1}{p}} < \infty \};$$

$$L_p(\nabla_\nu) = \{ f \in L_0(\nabla_\nu) : \mathrm{P} f \mathrm{P}_p = (\int_\Omega | f |^p \, d\nu)^{\frac{1}{p}} = (\int_\Omega h \cdot | f |^p \, d\mu)^{\frac{1}{p}} < \infty \}.$$

In this case the map $U : L_p(\nabla_\mu) \to L_p(\nabla_\nu)$, defined by the following equality

$$U(f) = h^{-\frac{1}{p}} f,\ f \in L_p(\nabla_\mu),$$

is the non trivial surjective isometry from $L_p(\nabla_\mu)$ onto $L_p(\nabla_\nu)$.

Bellow we show that this statement is not true for F-spaces $L_{\log}(\nabla_\mu)$ and $L_{\log}(\nabla_\nu)$.

Note that

$$L_{\log}(\nabla_\nu) = \{ f \in L_0(\nabla) : \int_\Omega \log(1 + | f |) \, d\nu < +\infty \} =$$

$$= \{ f \in L_0(\nabla) : \int_\Omega h \cdot \log((1 + | f |) \, d\mu < +\infty \} =$$

$$= \{ f \in L_0(\nabla) : \int_\Omega \log((1 + | f |)^h) \, d\mu < +\infty \}$$

and

$$\mathrm{P} f \mathrm{P}_{log,\nu} = \int_\Omega log(1 + | f |) \, d\nu = \int_\Omega h \cdot \log((1 + | f |) \, d\mu < +\infty \} =$$

$$= \int_\Omega \log((1 + | f |)^h) \, d\mu.$$

Let ∇ be a non-atomic complete Boolean algebra that is, a Boolean algebra ∇ has not atoms. Let $\nabla_e = \{g \in \nabla : g \le e\}$, where $0 \ne e \in \nabla$. By $\tau(\nabla_e)$ denote the minimal cardinality of a set that is dense in ∇_e with respect to the order topology ((o)-topology). The non-atomic complete Boolean algebra ∇ is said to be *homogeneous* if $\tau(\nabla_e) = \tau(\nabla_g)$ for any nonzero $e, g \in \nabla$. The cardinality $\tau(\nabla)$ is called the *weight* of the homogeneous Boolean algebra ∇ (see, for example [7, chapter VII]).

Theorem 3.1 *Let ∇ be a complete homogeneous Boolean algebra, μ and ν be finite strictly positive measures on ∇. $L_{log}(\nabla_\mu)$ is isometric to $L_{log}(\nabla_\nu)$ iff* $\dfrac{\int_\Omega h d\mu}{\mu(\Omega)} = 1.$

Proof. Necessity is proved in [5], see Theorem 5. We prove sufficiency. Let $\dfrac{\int_\Omega h d\mu}{\mu(\Omega)} = 1$. Then using the equality $\int_\Omega h d\mu = \int_\Omega h^{-1} h d\nu = \nu(\Omega)$, where $h^{-1} = \dfrac{d\mu}{d\nu}$, we obtain $\nu(\Omega) = \mu(\Omega)$. Hence, there is a measure-preserving automorphism α from ∇_μ onto ∇_ν, i.e. $\mu(e) = \nu(\alpha(e))$ for any $e \in \nabla_\mu$ ([7, chapter VII, Theorems 5 and 6]). Denote by J_α the isomorphism of the algebra $L_0(\nabla_\mu) = L_0(\nabla_\nu)$ such that $J_\alpha(e) = \alpha(e)$ for all $e \in \nabla_\mu$. That's why for any $f \in L_{log}(\nabla_\mu)$, we have from ([5], Proposition 3)

$$\| f \|_{log,\mu} = \int_\Omega log(1+|f(\omega)|) d\mu = \int_\Omega J_\alpha(log(1+|f(\omega)|)) d\nu =$$

$$= \int_\Omega log(1+J_\alpha|f(\omega)|) d\nu = \| J_\alpha |f(\omega)| \|_{log,\nu}.$$

Hence, the J_α is a bijective linear isometry from $L_{log}(\nabla_\mu)$ onto $L_{log}(\nabla_\nu)$.

Let $h = \dfrac{d\nu}{d\mu}$ and

$\Omega_> = \{\omega \in \Omega : h(\omega) > 1\}, \Omega_< = \{\omega \in \Omega : h(\omega) < 1\}, \Omega_= = \{\omega \in \Omega : h(\omega) = 1\} = \Omega \setminus (\Omega_> \cup \Omega_<)$.

Denote:

$S_> = S_>(\Omega, h) = \dfrac{\int_{\Omega_>} (h(x)-1) d\mu}{\mu(\Omega_>)}$ and $S_< = S_<(\Omega, h) = \dfrac{\int_{\Omega_<} (1-h(x)) d\mu}{\mu(\Omega_<)}$. The following theorem establishes 5 conditions equivalent to the isometricity of F-spaces.

Theorem 3.2 *Let* ∇ *be a complete homogeneous algebra,* μ *and* ν *be finite strictly positive measures on* ∇, *then the following conditions are equivalent*

(i). $L_{log}(\nabla_\mu)$ and $L_{log}(\nabla_\nu)$ are isometric;

(ii). $\dfrac{\int_\Omega h(\omega)d\mu}{\mu(\Omega)} = 1;$

(iii). $\dfrac{\int_\Omega h^{-1}(\omega)d\nu}{\nu(\Omega)} = 1$, where $h^{-1} = \dfrac{d\mu}{d\nu}$;

(iv). $\nu(\Omega) = \mu(\Omega);$

(v). $S_> = S_<;$

(vi). there is a measure-preserving automorphism α from ∇_μ onto ∇_ν.

Proof. $(i) \Leftrightarrow (ii)$ follows from Theorem 1.

$(ii) \Leftrightarrow (iii)$ $\dfrac{\int_\Omega h^{-1}(\omega)d\nu}{\nu(\Omega)} = \dfrac{\int_\Omega h(\omega)h^{-1}(\omega)d\mu}{\int_\Omega d\nu} = \dfrac{\mu(\Omega)}{\int_\Omega h(\omega)d\mu} = 1.$ The reverse implication is proved similarly.

$(iii) \Leftrightarrow (iv)$ $\dfrac{\int_\Omega h^{-1}(\omega)d\nu}{\nu(\Omega)} = \dfrac{\int_\Omega h(\omega)h^{-1}(\omega)d\mu}{\int_\Omega d\nu} = \dfrac{\mu(\Omega)}{\nu(\Omega)} = 1 \Leftrightarrow \mu(\Omega) = \nu(\Omega)$

$(iv) \Leftrightarrow (vi)$ follows [8, chapter VII, Theorems 5 and 6]

$(ii) \Leftrightarrow (v)$

$$0 = \frac{\int_\Omega h(\omega)d\mu}{\mu(\Omega)} - 1 = \frac{\int_\Omega h(\omega)d\mu}{\mu(\Omega)} - \frac{\int_\Omega d\mu}{\mu(\Omega)} = \frac{\int_\Omega (h(\omega)-1)d\mu}{\mu(\Omega)} = \frac{\int_{\Omega_>} (h(\omega)-1)d\mu}{\mu(\Omega_>)} + \frac{\int_{\Omega_<} (h(\omega)-1)d\mu}{\mu(\Omega_<)} + \frac{\int_{\Omega_=} (h(\omega)-1)d\mu}{\mu(\Omega_=)} = S_> - S_< \Leftrightarrow S_> = S_<$$

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